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Enumeration of k-plane trees and forests

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Abstract

A k-plane tree is an ordered tree in which the vertices are labelled by integers $\{1, 2, \ldots, k\}$ and satisfies the condition $i + j \leqslant k + 1$ where i and j are adjacent vertices in the tree. These trees are known to be counted by Fuss-Catalan numbers. In this paper, we use generating functions and decomposition of trees to enumerate these trees according to degree of the root, label of the first child of the root and number of forests of k-plane trees. The results of this paper generalize known results for 2-plane trees and plane trees.

Keywords: k-plane tree, degree, first child, forest.

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1. Introduction

In the last two decades, plane trees have been generalized by assigning labels to the vertices of the trees. Plane trees are rooted trees drawn in the plane such that all subtrees are ordered. They are one of the many structures counted by the famous Catalan numbers [14]. Thus, the number of plane trees on n vertices is given by

$$\frac{1}{n}\binom{2n-2}{n-1} \tag{1.1}$$

(See [1]). The level of a vertex v in a tree T, is defined as the number of edges in the path from the root of T to v. The root is therefore at level 0. A vertex i at level ℓ is a child (resp. parent) of vertex j if i and j are adjacent and j resides at level $\ell - 1$ (resp. level $\ell + 1$). The vertices that share a parent are called siblings and the one that appears on the far left is the first child of the parent. The number of children of a vertex is its outdegree. A vertex of outdegree zero is a leaf while a vertex of outdegree at least one is an internal vertex. The degree of a vertex u is the number of edges that are incident to u. Note that for the root, degree and outdegree coincide. The formulas for the number of plane trees with a given number of leaves, root degree, number of vertices of a given degree that reside at a certain level and the total number

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of vertices of a given degree were obtained by the authors of [1]. A forest is a collection of trees and it is a way of generalizing trees. In [15], it is showed that the number of forests of plane trees on n vertices with r components is given by

$$\frac{r}{n}\binom{2n-r-1}{n-r}. \tag{1.2}$$

Note that if we set $r = 1$ in (1.2), we obtain (1.1). Seo and Shin [13] obtained, by means of reverse Prüfer sequence introduced earlier by the same authors in [12], that the number of forests of plane trees on n vertices with r components in which roots of the components are labelled with integers $a_1, a_2, \dots, a_r$ where $a_1 < a_2 < \cdots < a_r$ is given by

$$r\binom{n}{r}\frac{(2n-r-1)!}{n!}.$$

In 2009, Gu and Prodinger [3] generalized plane trees by introducing another class of combinatorial structures, which they called 2-plane trees. These are plane trees in which the vertices are labelled 1 or 2 such that edges whose endpoints are both labelled 2 are not allowed. If the number of vertices is n and the root degree is labelled 1 and 2 then the number of these trees is given by

$$\frac{1}{n-1}\binom{3n-3}{n-2} \tag{1.3}$$

and

$$\frac{1}{n}\binom{3n-2}{n-1} \tag{1.4}$$

respectively. The total number of 2-plane trees on n vertices is obtained by adding (1.3) and (1.4) to obtain

$$\frac{2}{n}\binom{3n-3}{n-1}.$$

A significant amount of work has since been done on the set of 2-plane trees to enumerate them by root degree, label of the first child of the root and level of a vertex [5]. Bijections with other combinatorial structures were achieved in [9]. Of particular interest is the bijection with the set of 2-noncrossing increasing trees. Let (i, j) be an edge of a labelled rooted tree where i is closest to the root. This edge is an ascent if $i < j$. A 2-noncrossing increasing tree is a noncrossing tree (i.e., a tree drawn in the plane such that vertices are on the circumference of a circle and edges do not cross inside the circle) whose vertices are labelled 1 or 2 such that the sum of endpoints of edges do not exceed 3 and all the edges of the tree are ascents if they are considered as paths from the root.

Gu, Prodinger and Wagner in [4] extended the work of Gu and Prodinger [3] to introduce and enumerate k-plane trees. A k-plane tree is a plane tree in which vertices receive labels from the set $\{1, 2, \dots, k\}$ such that the sum of labels of any adjacent vertices is at most $k+1$. They proved that the number of these trees on n vertices with root labelled a is

$$\frac{k-a+1}{kn-a+1}\binom{(k+1)n-a-1}{n-1}. \tag{1.5}$$

Summing over all a in (1.5), we find that the total number of k-plane trees on n vertices is

$$\frac{k}{n}\binom{(k+1)(n-1)}{n-1}.$$

The authors also constructed a bijection between the set of k-plane trees with root labelled k and the set of $(k+1)$-ary trees. A d-ary tree is a plane tree in which every vertex has at most d children.

In 2023, Okoth and Wagner [10] used symbolic method to enumerate k-plane trees by the number of occurrence of labels of a certain type. They thus reproved the enumerative formulas obtained by Gu, Prodinger and Wagner in [4]. A k-noncrossing tree is a noncrossing tree which receives labels from the set $\{1, 2, \ldots, k\}$ such that if (i, j) is an ascent in a path from the root (always vertex 1) then $i + j \leqslant k + 1$. If there are only ascents in the k-noncrossing tree, then we get a k-noncrossing increasing tree. Recently, in 2024, Nyariaro and Okoth [6] constructed a bijection between the set of $2k$-plane trees and the set of k-noncrossing trees, both on n vertices. Earlier, Okoth [7] constructed a bijection between the set of k-plane trees on n vertices and the set of k-noncrossing increasing trees on n vertices. It is worth noting that in both bijections, the degrees of the root in all the trees are retained. Forests of k-noncrossing trees which satisfy certain conditions were enumerated by Okoth in [8].

In this paper, we use the following theorem to extract the coefficient of a variable in a given power series which satisfies a certain condition.

Theorem 1.1 (Lagrange Inversion Formula, [15, 16]). Let $p(x)$ be a generating function that satisfies the functional equation $p(x) = x\psi(p(x)$, where $\psi(0) \neq 0$. Then, we have

$$[x^n]p(x)^r = \frac{r}{n}[s^{n-r}]\psi(s)^n.$$

Theorem 1.2 (Binomial Theorem, [11]). Let a, b and n be integers, then

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}.$$

We also have [11]

$$\binom{-n}{j}(-1)^j = \binom{n+j-1}{j}.$$

Identity 1.3 (Hockey Stick Identity, [11]). Let r and k be positive integers, then

$$\sum_{j=r}^{k} \binom{j}{r} = \binom{k+1}{r+1}.$$

Identity 1.4 (Vandermonde Convolution, [11]). Let r, m and n be positive integers, then

$$\sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j} = \binom{m+n}{r}.$$

The following identity is a generalization of Vandermonde Convolution:

Identity 1.5 (Rothe-Hagen Identity, [2]). Let a, b, m and z be positive integers, then

$$\sum_{c=0}^{m} \frac{a}{a+cz}\binom{a+cz}{c}\frac{b}{b+(m-c)z}\binom{b+(m-c)z}{m-c} = \frac{a+b}{a+b+mz}\binom{a+b+mz}{m}. \tag{1.6}$$

This paper is organized as follows. In Section 2, we enumerate k-plane trees by degree of the root. The result is related to other combinatorial structures, based on the bijection already obtained in [9]. The study is extended to obtain counting formulas for k-plane trees in which the label of the root and that of its first child are specified in Section 3. In Section 4, we obtain formulas which count the number of forests of k-plane trees with a given number of vertices and labels of the components. The paper is concluded in Section 5 by giving a summary of results contained herein and ways in which this work could be extended.

2. Enumeration by root degree

In the sequel, we obtain our first result:

Theorem 2.1. Let $P_k(n, i, j, d)$ be the set of all k-plane trees on n vertices whose root is labelled by i such that all d children of the root are labelled j where $i + j \leqslant k + 1$. Then

$$|P_k(n, i, j, d)| = \frac{d(k - j + 1)}{(k + 1)(n - 1) - dj} \binom{(k + 1)(n - 1) - dj}{n - d - 1}. \tag{2.1}$$

Remark 2.2. We remark that (2.1) is independent of the label of the root, i.e., $|P_k(n, r, j, d)| = |P_k(n, s, j, d)|$ whenever $r, s \leqslant k - j + 1$.

Setting $k = 1$ and $j = 1$ in (2.1), we recover the formula

$$\frac{d}{2n - d - 2} \binom{2n - d - 2}{n - d - 1}$$

for the number of plane trees on n vertices with root of degree d obtained by Dershowitz and Zaks [1] in 1980. Moreover, setting $k = 2$ and $j = 1$ (resp. $k = 2$ and $j = 2$) we get the formulas

$$\frac{2d}{3n - d - 3} \binom{3n - d - 3}{n - d - 1} \tag{2.2}$$

and

$$\frac{d}{3n - 2d - 3} \binom{3n - 2d - 3}{n - d - 1}$$

for the number of 2-plane trees on n vertices with root labelled by 1 such that all children of the root are labelled by 1 and 2 respectively. These results were obtained by Lumumba, Okoth and Kasyoki in [5]. In the same paper, it was proved that Formula (2.2) counts 2-plane trees on n vertices with root labelled 2. This actually follows from Remark 2.2. Let us now prove Theorem 2.1.

Proof of Theorem 2.1. Let $P_i(x)$ be the generating function for the number of k-plane trees with root labelled i for $i = 1, 2, \ldots, k$. Here, x marks a vertex. Since an edge with one of its endpoints labelled i must have the other endpoint labelled with a positive integer less than or equal to $k - i + 1$, then

$$P_i(x) = \frac{x}{1 - P_1(x) - \cdots - P_{k-i+1}(x)}. \tag{2.3}$$

As noted by Gu, Prodinger and Wagner in [4], the system of functional equations (2.3) is easily solved by letting

$$P_i(x) = \frac{y}{(1 + y)^i} \text{ and } x = \frac{y}{(1 + y)^{k+1}}. \tag{2.4}$$

We have,

$$\begin{aligned} P_i(x) &= \frac{x}{1 - P_1(x) - \cdots - P_{k-i+1}(x)} = \frac{x}{1 - \sum_{j=0}^{k-i+1} \frac{y}{(1+y)^j}} = \frac{x}{1 - (1 - (1 + y)^{i-k-1})} \\ &= \frac{x}{(1 + y)^{i-k-1}}. \end{aligned}$$

Substituting $x = \frac{y}{(1+y)^{k+1}}$, we get

$$P_i(x) = \frac{y}{(1 + y)^{k+1}} \cdot \frac{1}{(1 + y)^{i-k-1}} = \frac{y}{(1 + y)^i}.$$

This proves that $P_i(x) = \frac{y}{(1+y)^i}$ and $x = \frac{y}{(1+y)^{k+1}}$ are the right substitutions for solving the system of the functional equations. So,

$$y = x(1+y)^{k+1} \tag{2.5}$$

and it is in a form we can apply the Lagrange Inversion Formula to extract the coefficient of x^m for any integer m.

Now, consider a k-plane in which the root is labelled i and has d children of label j as shown in Figure 1.

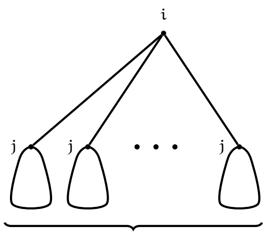

Figure 1: A k plane tree with root labelled i and d children of the root labelled j.

By the decomposition of the k-plane tree, we need to extract the coefficient of x^n in $xP_j(x)^d$:

$$[x^n]xP_j(x)^d = [x^{n-1}]P_j(x)^d = [x^{n-1}](y(1+y)^{-j})^d = [x^{n-1}]y^d(1+y)^{-dj}.$$

By Binomial Theorem, we get

$$[x^n]xP_j(x)^d = [x^{n-1}]y^d \sum_{h\geqslant 0} \binom{-dj}{h} y^h = \sum_{h\geqslant 0} \binom{-dj}{h} [x^{n-1}]y^{h+d}.$$

Applying Lagrange Inversion Formula with y given in (2.5), we have

$$\begin{aligned}
[x^n]xP_j(x)^d &= \sum_{h\geqslant 0} \binom{-dj}{h} \frac{h+d}{n-1} [t^{n-h-d-1}](1+t)^{(k+1)(n-1)} \\
&= \sum_{h\geqslant 0} \binom{-dj}{h} \frac{h+d}{n-1} \binom{(k+1)(n-1)}{n-h-d-1} \\
&= \frac{d}{n-1} \sum_{h\geqslant 0} \left[\binom{-dj}{h} - j\binom{-dj-1}{h-1} \right] \binom{(k+1)(n-1)}{n-h-d-1}.
\end{aligned}$$

Applying Vandermonde Convolution, we get

$$\begin{aligned}
[x^n]xP_j(x)^d &= \frac{d}{n-1} \left[\binom{(k+1)(n-1)-dj}{n-d-1} - j\binom{(k+1)(n-1)-dj-1}{n-d-2} \right] \\
&= \frac{d}{n-1} \left[1 - \frac{j(n-d-1)}{(k+1)(n-1)-dj} \right] \binom{(k+1)(n-1)-dj}{n-d-1} \\
&= \frac{d}{n-1} \left[\frac{(n-1)(k+1-j)}{(k+1)(n-1)-dj} \right] \binom{(k+1)(n-1)-dj}{n-d-1} \\
&= \frac{d(k-j+1)}{(k+1)(n-1)-dj} \binom{(k+1)(n-1)-dj}{n-d-1}
\end{aligned}$$

which is the desired result. □

By setting $j = 1$ in (2.1) and making use of Remark 2.2, we find that:

Corollary 2.3. There are

$$\frac{d}{n-1}\binom{(k+1)(n-1)-d-1}{n-d-1} \tag{2.6}$$

k-plane trees on n vertices whose root is labelled k and has degree d.

Based on bijections constructed by Okoth in [7] and by Nyariaro and Okoth in [6], we find that:

(i) Formula (2.6) also counts the number of noncrossing increasing trees on n vertices such that the root is labelled k and is of degree d.

(ii) Formula (2.6) Gives the number of Dyck paths of semilength $n-1$ such that the up-steps are labelled with integers from the set $\{1, 2, \dots, k+1\}$, the sum of labels of two adjacent up-steps is no more than $k+1$, there are d returns to the x-axis and the d up-steps incident to the x-axis are labelled 1.

(iii) The number of k-noncrossing trees on n vertices with root labelled k and of degree d is given by

$$\frac{d}{n-1}\binom{(2k+1)(n-1)-d-1}{n-d-1}.$$

This is because the set of $2k$-plane trees on n vertices is in bijection with the set of k-noncrossing trees. The bijection preserves root degree. The formula was also obtained by Okoth in [8] by constructing a bijection with the set of lattice paths which lie weakly below the line $y = kx$.

Summing over all d in (2.6), we get the following corollary.

Corollary 2.4. The total number of k-plane trees on n vertices with roots labelled k is given by

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}. \tag{2.7}$$

Proof. As already been mentioned, we sum over d in (2.6), i.e.,

$$\begin{aligned}\sum_{d=1}^{n-1}\frac{d}{n-1}\binom{(k+1)(n-1)-d-1}{k(n-1)-1} &= \sum_{d=1}^{n-1}\left[\frac{(k+1)(n-1)}{n-1}-\frac{((k+1)(n-1)-d)k}{k(n-1)}\right]\binom{(k+1)(n-1)-d-1}{k(n-1)-1}\\ &= \sum_{d=1}^{n-1}(k+1)\binom{(k+1)(n-1)-d-1}{k(n-1)-1} - \sum_{d=1}^{n-1}k\binom{(k+1)(n-1)-d}{k(n-1)}.\end{aligned}$$

The first equality follows by telescoping. Now by change of sum limits we get,

$$\sum_{d=1}^{n-1}\frac{d}{n-1}\binom{(k+1)(n-1)-d-1}{k(n-1)-1} = \sum_{j=k(n-1)-1}^{(k+1)(n-1)-2}(k+1)\binom{j}{k(n-1)-1} - \sum_{j=k(n-1)}^{(k+1)(n-1)-1}k\binom{j}{k(n-1)}.$$

By Hockey-Stick Identity, we obtain

$$\begin{aligned}\sum_{d=1}^{n-1}\frac{d}{n-1}\binom{(k+1)(n-1)-d-1}{k(n-1)-1} &= (k+1)\binom{(k+1)(n-1)-1}{k(n-1)} - k\binom{(k+1)(n-1)}{k(n-1)+1}\\ &= \left[\frac{k(n-1)+1}{n-1}-k\right]\binom{(k+1)(n-1)}{n-2}\\ &= \frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}.\end{aligned}$$

Hence the proof. □

Theorem 2.5. There are

$$\frac{d(k+1)-s}{(k+1)(n-1)-s}\binom{(k+1)(n-1)-s}{n-d-1}\binom{d}{d_1, d_2, \dots, d_{k-i+1}} \tag{2.8}$$

k-plane trees on n vertices whose root is labelled i and of degree d such that there are d_j children of the root labelled j for $j = 1, 2, 3, \dots, k-i+1$. Here, $s := d_1 + 2d_2 + \cdots + (k-i+1)d_{k-i+1}$.

Proof. Let $P_j(x)$ be the generating function for the number of k-plane trees rooted at a vertex of label j. Since there are d_j subtrees rooted at the children of the root for $j = 1, 2, \dots, k$, the desired generating function for the number of trees in the statement of the theorem is $xP_1(x)^{d_1}P_2(x)^{d_2}\cdots P_{k-i+1}(x)^{d_{k-i+1}}$. We extract the coefficient of x^n in the generating function.

$$\begin{aligned}[x^n]xP_1(x)^{d_1}P_2(x)^{d_2}\cdots P_{k-i+1}(x)^{d_{k-i+1}} &= [x^{n-1}]\left(\frac{y}{1+y}\right)^{d_1}\cdot\left(\frac{y}{(1+y)^2}\right)^{d_2}\cdots\left(\frac{y}{(1+y)^{k-i+1}}\right)^{d_{k-i+1}}\\ &= [x^{n-1}]y^{d_1+d_2+\cdots+d_{k-i+1}}(1+y)^{-(d_1+2d_2+\cdots+(k-i+1)d_{k-i+1})}\end{aligned}$$

where y is as given in (2.5). Since the root has degree d then $d = d_1 + d_2 + \cdots + d_{k-i+1}$. Also, $s = d_1 + 2d_2 + \cdots + (k-i+1)d_{k-i+1}$ so that

$$[x^n]xP_1(x)^{d_1}P_2(x)^{d_2}\cdots P_{k-i+1}(x)^{d_{k-i+1}} = [x^{n-1}]y^d(1+y)^{-s}.$$

By Binomial Theorem, we have

$$[x^{n-1}]y^d(1+y)^{-s} = [x^{n-1}]y^d\sum_{h\geqslant 0}\binom{-s}{h}y^h = \sum_{h\geqslant 0}\binom{-s}{h}[x^{n-1}]y^{h+d}.$$

By Lagrange Inversion Formula, we obtain

$$\begin{aligned}[x^{n-1}]y^d(1+y)^{-s} &= \sum_{h\geqslant 0}\binom{-s}{h}\frac{h+d}{n-1}[t^{n-h-d-1}](1+t)^{(k+1)(n-1)}\\ &= \sum_{h\geqslant 0}\binom{-s}{h}\frac{h+d}{n-1}\binom{(k+1)(n-1)}{n-h-d-1}\\ &= \frac{1}{n-1}\sum_{h\geqslant 0}(h+d)\binom{-s}{h}\binom{(k+1)(n-1)}{n-h-d-1}\\ &= \frac{1}{n-1}\sum_{h\geqslant 0}\left[d\binom{-s}{h} - s\binom{-s-1}{h-1}\right]\binom{(k+1)(n-1)}{n-h-d-1}.\end{aligned}$$

By Vandermonde Convolution, we get

$$\begin{aligned}[x^{n-1}]y^d(1+y)^{-s} &= \frac{1}{n-1}\left[d\binom{(k+1)(n-1)-s}{n-d-1} - s\binom{(k+1)(n-1)-s-1}{n-d-2}\right]\\ &= \frac{1}{n-1}\left[d\binom{(k+1)(n-1)-s}{n-d-1} - \frac{s(n-d-1)}{(k+1)(n-1)-s}\binom{(k+1)(n-1)-s}{n-d-1}\right]\\ &= \frac{1}{n-1}\left[d - \frac{s(n-d-1)}{(k+1)(n-1)-s}\right]\binom{(k+1)(n-1)-s}{n-d-1}\\ &= \frac{1}{n-1}\left(\frac{d[(k+1)(n-1)-s]-s(n-d-1)}{(k+1)(n-1)-s}\right)\binom{(k+1)(n-1)-s}{n-d-1}\\ &= \frac{1}{n-1}\left(\frac{(n-1)[d(k+1)-s]}{(k+1)(n-1)-s}\right)\binom{(k+1)(n-1)-s}{n-d-1}\\ &= \frac{d(k+1)-s}{(k+1)(n-1)-s}\binom{(k+1)(n-1)-s}{n-d-1}.\end{aligned}$$

Now, there are

$$\binom{d}{d_1, d_2, \ldots, d_{k-i+1}}$$

ways of assigning labels to the children of the root so that there are d_j children labelled j for $j = 1, 2, \ldots, k - i + 1$. By the product rule of counting the result follows. □

We obtain Theorem 2.1, by setting $s = jd$ and $d_r = 0$ for all $r \neq j$. If $k = 1$ in Theorem 2.5 then $i = 1$, $s = d$, $d_1 = d$ and $d_r = 0$ for all $r = 2, 3, \ldots, k - i + 1$. This implies that there are

$$\frac{d}{2n - d - 2}\binom{2n - d - 2}{n - d - 1}$$

plane trees on n vertices with root degree d.

If $k = 2$ and $i = 1$ in (2.8) then $d_1 + d_2 = d$ and $d_1 + 2d_2 = s$. This means that $d_2 = d - d_1$ and $s = 2d - d_1$. We thus obtain that there are

$$\frac{d + d_1}{3n - 2d + d_1 - 3}\binom{3n - 2d + d_1 - 3}{n - d - 1}\binom{d}{d_1} \tag{2.9}$$

2-plane trees on n vertices with root labelled 1 and has d children of which d_1 are labelled 1. This result was also obtained by authors of [5]. Summing over all d_1 and d in (2.9), we find the total number of 2-plane trees on n vertices with root labelled 1.

If $k = 2$ and $i = 2$ in (2.8) then $j = 1$, $d_1 = d$ and $s = d$. It follows that there

$$\frac{2d}{3n - d - 3}\binom{3n - d - 3}{n - d - 1} \tag{2.10}$$

2-plane trees on n vertices with root labelled 2 and has d children labelled 1. Summing over all d in (2.10), we find the total number of 2-plane trees on n vertices with root labelled 2. Setting $d = 1$ and $d_r = 0$ in Equation (2.8) for all $r \neq j$, we find that $d_j = 1$ and $s = j$. So, the number of planted k-plane trees on n vertices such that the root is labelled j is given by

$$\frac{k + 1 - j}{(k + 1)(n - 1) - j}\binom{(k + 1)(n - 1) - j}{n - 2}$$

and by summing over all j, we get the total number of planted k-plane trees on $n > 1$ vertices as

$$\frac{k}{n - 1}\binom{(k + 1)(n - 2)}{n - 2}.$$

3. Enumeration by label of the first child of the root

In this section, we are interested in obtaining counting formulas for k-plane trees given the labels of the root and its first child. A similar study with $k = 2$ has been done by Lumumba, Okoth and Kasyoki in [5].

Theorem 3.1. The set of k-plane trees on $n > 1$ vertices with root labelled i and first child of the root labelled j is enumerated by

$$\frac{2k - i - j + 2}{(k + 1)n - i - j}\binom{(k + 1)n - i - j}{n - 2}. \tag{3.1}$$

Proof. Based on the decomposition of k-plane trees given in Figure 2, the theorem is proved by extracting the coefficient of x^n in $P_i(x)P_j(x)$ which we shall now perform:

$$[x^n]P_i(x)P_j(x) = [x^n]\frac{y}{(1 + y)^i} \cdot \frac{y}{(1 + y)^j} = [x^n]y^2(1 + y)^{-(i+j)}.$$

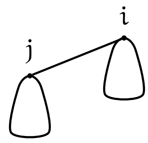

Figure 2: Decomposition of k-plane trees with root labelled i and first child of the root is labelled j.

We have used the substitutions given in (2.4). By Binomial Theorem, we get

$$[x^n]P_i(x)P_j(x) = [x^n]y^2 \sum_{a\geqslant 0} \binom{-i-j}{a} y^a = \sum_{a\geqslant 0} \binom{-i-j}{a} [x^n]y^{a+2}.$$

By Lagrange Inversion Formula,

$$\begin{aligned}
[x^n]P_i(x)P_j(x) &= \sum_{a\geqslant 0} \frac{a+2}{n}\binom{-i-j}{a}[t^{n-a-2}](1+t)^{(k+1)n} \\
&= \sum_{a\geqslant 0} \frac{a+2}{n}\binom{-i-j}{a}\binom{(k+1)n}{n-a-2} \\
&= \frac{1}{n}\sum_{a\geqslant 0}\left[2\binom{-i-j}{a} - (i+j)\binom{-i-j-1}{a-1}\right]\binom{(k+1)n}{n-a-2}.
\end{aligned}$$

Applying Vandermonde Convolution, we obtain

$$\begin{aligned}
[x^n]P_i(x)P_j(x) &= \frac{1}{n}\left[2\binom{(k+1)n-i-j}{n-2} - (i+j)\binom{(k+1)n-i-j-1}{n-3}\right] \\
&= \frac{1}{n}\left[2 - \frac{(i+j)(n-2)}{(k+1)n-i-j}\right]\binom{(k+1)n-i-j}{n-2} \\
&= \frac{1}{n}\left[\frac{n(2(k+1)-i-j)}{(k+1)n-i-j}\right]\binom{(k+1)n-i-j}{n-2} \\
&= \frac{2k-i-j+2}{(k+1)n-i-j}\binom{(k+1)n-i-j}{n-2}.
\end{aligned}$$

□

By setting $i+j=r$ in (3.1), we find that there are

$$\frac{2k-r+2}{(k+1)n-r}\binom{(k+1)n-r}{n-2} \tag{3.2}$$

k-plane trees on n vertices with such that the sum of the labels of the root and its first child is r. The maximum value r can take is $k+1$ and the number of trees in which the sum of the labels of the root and its first child is r is given by

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}.$$

This result is achieved by setting $r=k+1$ in (3.2). We obtain further simple case results.

(i) If $k=1$ and $i=j=1$ in (3.1), we obtain the $(n-1)^{\text{th}}$ Catalan number as the number of plane (i.e., 1-plane) trees on n vertices.

(ii) Setting $r = 3$ and $k = 2$ in (3.2), we get

$$\frac{1}{n-1}\binom{3n-3}{n-2}$$

as the counting formula for the number of 2−plane trees with root labelled 1(resp. 2) on $n > 1$ vertices such that the first child of the root is labelled 2 (resp. 1). This result was also obtained by the authors of [5].

(iii) Putting $r = 2i$ in (3.2) or $j = i$ in (3.1) we get

$$\frac{2(k-i+1)}{(k+1)n-2i}\binom{(k+1)n-2i}{n-2}, \tag{3.3}$$

which counts k-plane trees on n vertices in which both the root and its first child are labelled i and $ineqk$. Setting $i = 1$ and $k = 2$ in (3.3) we find that there are

$$\frac{2}{n}\binom{3n-3}{n-2},$$

2−plane trees on $n > 1$ vertices with both the root and its first child labelled 1. We remark that (3.3) also counts k-plane trees on $n+1$ vertices in which the root is labelled 1 such that the root is of degree 2 and all children of the root are labelled i as proved in Theorem 2.1. This can be shown by a simple counting argument: Delete the edge connecting the root to its first child and connecting the root and its first child to a new root in such a way that the subtree rooted at the initial root is on the right. The reverse procedure is easily seen, i.e., delete the root and all its incident edges.

Theorem 3.2. There are

$$\frac{k(r+2)-i-j+2}{(k+1)n-r-i-j}\binom{(k+1)n-r-i-j}{n-r-2} \tag{3.4}$$

k-plane trees on $n > 1$ vertices with the root labelled i such that the first r children of the root are labelled 1 and the $(r+1)^{th}$ child of the root is labelled j.

Proof. Consider a k-plane tree T on n vertices such that the root is labelled i such that the first r children of the root are labelled 1 and the $(r+1)^{th}$ child of the root is labelled j. Let T_1 be the subtree of T which comprises of the root and all the subtrees rooted at the first r children of the root. Also let T_2 be the tree obtain when you delete the subtrees rooted at the first r subtrees of T. The subtree T_2 will thus have the first child of the root labelled j. This decomposition is given in Figure 3. Let the number of vertices of T_1

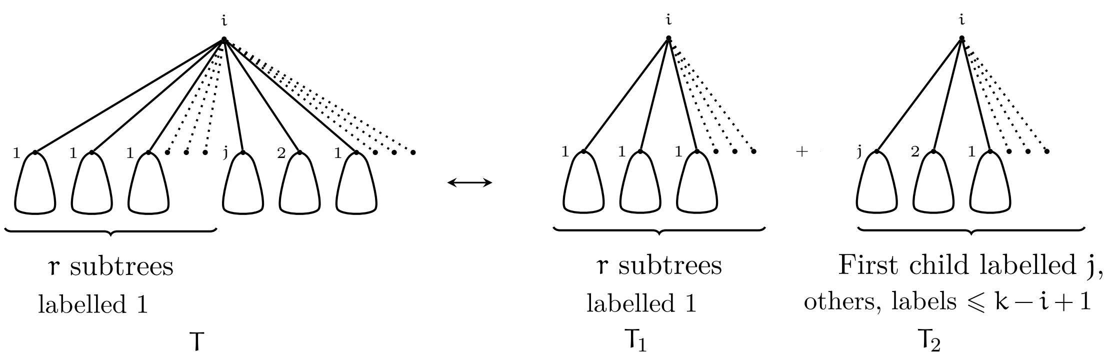

Figure 3: Decomposition of k-plane tree with root labelled 1, first d children of the root labelled r and the $(r+1)^{th}$ child labelled j.

be m and the number of these trees is given by

$$\frac{rk}{(k+1)(m-1)-r}\binom{(k+1)(m-1)-r}{m-r-1} \tag{3.5}$$

obtained by setting $n=m, d=r$ and $j=1$ in (2.1). We also have that T_2 has $n-m+1$ vertices and the number of these trees is obtained by setting $n=n-m+1$ in (3.1).

$$\frac{2k-i-j+2}{(k+1)(n-m+1)-i-j}\binom{(k+1)(n-m+1)-i-j}{n-m-1}. \tag{3.6}$$

By the product and sum rules of counting, the desired result obtained by taking the product of (3.5) and (3.6) and summing over all m, i.e.,

$$\begin{aligned}
&\sum_{m=r+1}^{n-1}\frac{rk}{(k+1)(m-1)-r}\binom{(k+1)(m-1)-r}{m-r-1}\frac{2k-i-j+2}{(k+1)(n-m+1)-i-j}\binom{(k+1)(n-m+1)-i-j}{n-m-1}\\
&=\sum_{a=0}^{n-r-2}\frac{rk}{(k+1)(a+r)-r}\binom{(k+1)(a+r)-r}{a}\frac{2k-i-j+2}{(k+1)(n-a-r)-i-j}\binom{(k+1)(n-a-r)-i-j}{n-a-r-2}\\
&=\sum_{a=0}^{n-r-2}\left[\frac{rk}{rk+a(k+1)}\binom{rk+a(k+1)}{a}\frac{2k-i-j+2}{2k-i-j+2+(k+1)(n-a-r-2)}\right.\\
&\qquad\qquad\left.\times\binom{2k-i-j+2+(k+1)(n-a-r-2)}{n-a-r-2}\right].
\end{aligned}$$

Using Rothe-Hagen Identity (1.6) with $a=2k-i-j+2, b=rk, c=n-a-r-2, m=n-r-2$ and $z=k+1$, we obtain that the number of trees in the statement of the theorem is

$$\begin{aligned}
&\frac{k(r+2)-i-j+2}{k(r+2)-i-j+2+(k+1)(n-r-2)}\binom{k(r+2)-i-j+2+(k+1)(n-r-2)}{n-r-2}\\
&\qquad=\frac{k(r+2)-i-j+2}{(k+1)n-r-i-j}\binom{(k+1)n-r-i-j}{n-r-2}.
\end{aligned}$$

This completes the proof. □

Alternative Proof of Theorem 3.2. The trees on the statement of the theorem can be decomposed as showed in Figure 4. Based on the decomposition, the generating function for these trees is therefore $P_1(x)^r P_j(x) P_i(x)$.

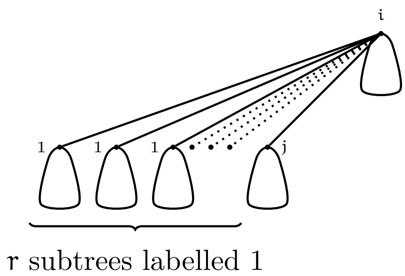

Figure 4: A decomposition of k-plane tree with root labelled 1, first d children of the root labelled r and the $(r+1)^{\text{th}}$ child labelled j.

Now, extract the coefficient of x^n in the generating function. Upon use of Binomial Theorem, we get

$$\begin{aligned}
[x^n]P_1(x)^r P_j(x) P_i(x) &= \left(\frac{y}{1+y}\right)^r\cdot\frac{y}{(1+y)^j}\cdot\frac{y}{(1+y)^i} = [x^n]y^{r+2}(1+y)^{-(r+i+j)}\\
&=[x^n]\sum_{s\geqslant 0}\binom{-r-i-j}{s}y^{r+s+2}.
\end{aligned}$$

By Lagrange Inversion Formula, we have

$$\begin{aligned}[x^n]P_1(x)^r P_j(x)P_i(x) &= \sum_{s\geqslant 0}\binom{-r-i-j}{s}[t^{n-r-s-2}]\frac{r+s+2}{n}(1+t)^{(k+1)n}\\ &= \sum_{s\geqslant 0}\binom{-r-i-j}{s}\frac{r+s+2}{n}\binom{(k+1)n}{n-r-s-2}.\end{aligned}$$

We make use of Vandermonde Convolution to obtain

$$\begin{aligned}[x^n]P_1(x)^r P_j(x)P_i(x) &= \sum_{s\geqslant 0}\binom{-r-i-j}{s}\frac{r+s+2}{n}\binom{(k+1)n}{n-r-s-2}\\ &= \frac{1}{n}\sum_{s\geqslant 0}\left[(r+2)\binom{-r-i-j}{s}-(r+i+j)\binom{-r-i-j-1}{s-1}\right]\binom{(k+1)n}{n-r-s-2}\\ &= \frac{1}{n}\left[(r+2)\binom{(k+1)n-r-i-j}{n-r-2}-(r+i+j)\binom{(k+1)n-r-i-j-1}{n-r-3}\right]\\ &= \frac{k(r+2)-i-j+2}{(k+1)n-r-i-j}\binom{(k+1)n-r-i-j}{n-r-2}.\end{aligned}$$

□

By setting $r=0$ in (3.4), we recover Formula (3.1) for the number of k-plane trees on n vertices with roots labelled by i and first child of the root labelled by j for $i+j\leqslant k+1$.

4. Enumeration of forests of k-plane trees

We consider two types of labellings of the vertices of k-plane trees. A vertex of a k-plane tree is given labels from both the sets $\{1,2,\dots,n\}$ and $\{1,2,\dots,k\}$. A label received from $\{1,2,\dots,n\}$ will be referred to as vertex number and the one obtained from $\{1,2,\dots,k\}$ is vertex label. A k-plane tree is said to be labelled if the vertices are assigned vertex numbers. To avoid redundancies, we study labelled k-plane trees.

4.1. Forests of k-plane trees

Theorem 4.1. The number of labelled k-plane forests on n vertices with r components whose roots are labelled i is

$$\frac{n!r(k-i+1)}{(k+1)n-ri}\binom{(k+1)n-ri}{n-r}. \tag{4.1}$$

Proof. Since each of the r components is labelled i, we extract the coefficient of x^n in $P_i(x)^r$:

$$[x^n]P_i(x)^r = [x^n](y(1+y)^{-i})^r = [x^n]y^r(1+y)^{-ri}.$$

By Binomial Theorem, we get

$$[x^n]P_i(x)^r = [x^n]y^r\sum_{h\geqslant 0}\binom{-ri}{h}y^h = \sum_{h\geqslant 0}\binom{-ri}{h}[x^n]y^{h+r}.$$

Applying Lagrange Inversion Formula with y given in (2.5), we have

$$\begin{aligned}[x^n]P_i(x)^r &= \sum_{h\geqslant 0}\binom{-ri}{h}\frac{h+r}{n}[t^{n-h-r}](1+t)^{(k+1)n}\\ &= \sum_{h\geqslant 0}\binom{-ri}{h}\frac{h+r}{n}\binom{(k+1)n}{n-h-r}\\ &= \frac{r}{n}\sum_{h\geqslant 0}\left[\binom{-ri}{h}-i\binom{-ri-1}{h-1}\right]\binom{(k+1)n}{n-h-r}.\end{aligned}$$

Upon application of Vandermonde Convolution, we get

$$\begin{aligned}[x^n]P_i(x)^r &= \frac{r}{n}\left[\binom{(k+1)n-ri}{n-r} - i\binom{(k+1)n-ri-1}{n-r-1}\right]\\ &= \frac{r}{n}\left[1-\frac{i(n-r)}{(k+1)n-ri}\right]\binom{(k+1)n-ri}{n-r}\\ &= \frac{r}{n}\left[\frac{n(k+1-i)}{(k+1)n-ri}\right]\binom{(k+1)n-ri}{n-r}\\ &= \frac{r(k-i+1)}{(k+1)n-ri}\binom{(k+1)n-ri}{n-r}.\end{aligned}$$

Now, there are $n!$ ways to assign vertex numbers to the vertices of the k-plane tree. By product rule of counting, the result follows. □

Corollary 4.2. There are

$$(n-1)!r\binom{2n-r-1}{n-1}$$

labelled plane forests on n vertices with r components.

Proof. Set $k=i=1$ in (4.1). □

We find a generalization of Theorem 4.1:

Theorem 4.3. There are

$$n!\frac{r(k+1)-s}{(k+1)n-s}\binom{(k+1)n-s}{n-r}\binom{r}{r_1, r_2, \dots, r_{k-i+1}} \tag{4.2}$$

labelled k-plane forests on n vertices with r components such that roots of r_j components are labelled j for $j=1,2,3,\dots,k-i+1$. Here, $s := r_1+2r_2+\cdots+(k-i+1)r_{k-i+1}$.

Proof. We extract the coefficient of x^n in $P_1(x)^{r_1}P_2(x)^{r_2}\cdots P_{k-i+1}(x)^{r_{k-i+1}}$.

$$\begin{aligned}[x^n]P_1(x)^{r_1}P_2(x)^{r_2}\cdots P_{k-i+1}(x)^{r_{k-i+1}} &= [x^n]\left(\frac{y}{1+y}\right)^{r_1}\cdot\left(\frac{y}{(1+y)^2}\right)^{r_2}\cdots\left(\frac{y}{(1+y)^{k-i+1}}\right)^{r_{k-i+1}}\\ &= [x^n]y^{r_1+r_2+\cdots+r_{k-i+1}}(1+y)^{-(r_1+2r_2+\cdots+(k-i+1)r_{k-i+1})}\end{aligned}$$

where y is as given in (2.5). Since the number of components is r then $r = r_1+r_2+\cdots+r_{k-i+1}$. Also, $s = r_1+2r_2+\cdots+(k-i+1)r_{k-i+1}$ so that

$$[x^n]P_1(x)^{r_1}P_2(x)^{r_2}\cdots P_{k-i+1}(x)^{r_{k-i+1}} = [x^n]y^r(1+y)^{-s}.$$

By Binomial Theorem, we have

$$[x^n]y^r(1+y)^{-s} = [x^n]y^r\sum_{h\geqslant 0}\binom{-s}{h}y^h = \sum_{h\geqslant 0}\binom{-s}{h}[x^n]y^{h+r}.$$

By Lagrange Inversion Formula, we obtain

$$\begin{aligned}[x^n]y^r(1+y)^{-s} &= \sum_{h\geqslant 0}\binom{-s}{h}\frac{h+r}{n}[t^{n-h-r}](1+t)^{(k+1)n}\\ &= \sum_{h\geqslant 0}\binom{-s}{h}\frac{h+r}{n}\binom{(k+1)n}{n-h-r}\\ &= \frac{1}{n}\sum_{h\geqslant 0}\left[r\binom{-s}{h} - s\binom{-s-1}{h-1}\right]\binom{(k+1)n}{n-h-r}.\end{aligned}$$

By Vandermonde Convolution, we get

$$\begin{aligned}
[x^n]y^r(1+y)^{-s} &= \frac{1}{n}\left[r\binom{(k+1)n-s}{n-r} - s\binom{(k+1)n-s-1}{n-r-1}\right] \\
&= \frac{1}{n}\left[r\binom{(k+1)n-s}{n-r} - \frac{s(n-r)}{(k+1)n-s}\binom{(k+1)n-s}{n-r}\right] \\
&= \frac{1}{n}\left(\frac{r((k+1)n-s)-s(n-r)}{(k+1)n-s}\right)\binom{(k+1)n-s}{n-r} \\
&= \frac{1}{n}\left(\frac{n(r(k+1)-s)}{(k+1)n-s}\right)\binom{(k+1)n-s}{n-r} \\
&= \frac{r(k+1)-s}{(k+1)n-s}\binom{(k+1)n-s}{n-r}.
\end{aligned}$$

Now, there are

$$\binom{r}{r_1, r_2, \ldots, r_{k-i+1}}$$

ways of arranging the r components of the forest. There are $n!$ ways of assigning vertex numbers to the vertices of the forest. By the product rule of counting the result follows. □

We obtain Theorem 4.1, by setting $s = ri$ and $r_d = 0$ for all $d \neq i$. If $k = 1$ in Theorem 4.3 then $i = 1$, $s = r$, $r_1 = r$ and $r_d = 0$ for all $d = 2, 3, \ldots, k-i+1$. This implies that there are

$$\frac{n!r}{2n-r}\binom{2n-r}{n-r}$$

labelled plane forests on n vertices with r components, a result also given in Corollary 4.2.

If $k = 2$ and $i = 1$ in (4.2) then $r_1 + r_2 = r$ and $r_1 + 2r_2 = s$. This means that $r_2 = r - r_1$ and $s = 2r - r_1$. We thus obtain that there are

$$\frac{n!(r+r_1)}{3n-2r+r_1}\binom{3n-2r+r_1}{n-r}\binom{r}{r_1}$$

labelled 2-plane forests on n vertices with r components, r_1 of which are have roots labelled 1.

If $k = 2$ and $i = 2$ in (4.2) then $j = 1$, $r_1 = r$ and $s = r$. It follows that there are

$$(n-1)!r\binom{3n-r-1}{n-r}$$

labelled 2-plane forests on n vertices with r components whose roots are labelled 1.

4.2. Forests of k-noncrossing increasing trees

Since in [7], Okoth constructed a bijection between the set of k-plane trees and the set of k-noncrossing increasing trees both on n vertices, then in this subsection, we consider forests of k-noncrossing increasing trees with the following two properties:

- Each component is rooted at a vertex whose vertex number is smallest.
- The components are k-noncrossing increasing trees with the root labelled by k, and the components do not intersect each other.

The result obtained in this section, therefore suffices for the enumeration of forests of k-plane trees which satisfy the stated properties.

In proving the following result, we use generating functions and mimic the proof of Okoth in [8] in which he obtained a formula for the number of forests of k-noncrossing trees satisfying the aforementioned properties.

Theorem 4.4. There are

$$\frac{1}{(k+1)n-kr}\binom{n}{r-1}\binom{(k+1)n-kr}{n-r} \tag{4.3}$$

forests of k-noncrossing increasing trees on n vertices with r components.

Proof. Let P be the generating function for the components, i.e., k-noncrossing increasing trees with roots labelled by k. We decompose the forests according to components containing vertex number 1 (See Figure 5). If this component is on ν vertices then there are ν spaces that are to be filled by forests $H_1, H_2, \ldots, H_\nu$. Note that some of these components may be empty.

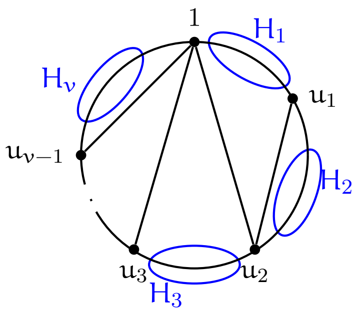

Figure 5: Decomposition of forests of k-noncrossing increasing tree according to vertex number 1.

So if $P(x) = \sum_{n\geqslant 1} p_n x^n$, then the generating function $H(x,z)$ for forests where z marks the number of components will satisfy

$$H(x,z) = 1 + z\sum_{\nu\geqslant 1} p_\nu x^\nu H(x,z)^\nu = 1 + zP(xH(x,z)).$$

Let $P(x) = x(1+y(x))$. Since $P^{k+1} - x^{k-1}P + x^k = 0$ (see [7]), then $y(x) = x(1+y(x))^{k+1}$. Now, let $H(x,z) = 1 + za$. Then, $1 + za = 1 + zP(x(1+za))$ or $a = P(x(1+za))$.

Define q as $q(t) = \frac{t}{P^{-1}(t)}$. Since $x(1+za) = P^{-1}(a)$, then $x(1+za) = \frac{a}{q(a)}$. This means that $a = x(1+za)q(a)$. Now by Lagrange Inversion Formula, the required formula is

$$[x^n z^r]H(x,z) = [x^n z^{r-1}]a = \frac{1}{n}[t^{n-1}z^{r-1}]((1+zt)q(t))^n = \frac{1}{n}\binom{n}{r-1}[t^{n-r}]q(t)^n.$$

To complete the proof, we need to obtain $[t^{n-r}]q(t)^n$. By definition, $t = P\left(\frac{t}{q(t)}\right)$.

We write $P(x) = x(1+y(x))$, to have

$$t = \frac{t}{q(t)}\left(1 + y\left(\frac{t}{q(t)}\right)\right)$$

or

$$q(t) = 1 + y\left(\frac{t}{q(t)}\right)$$

where $y(x)$ satisfies $y(x) = x(1+y(x))^{k+1}$. So,

$$q(t) - 1 = y\left(\frac{t}{q(t)}\right) = \frac{t}{q(t)}\left(1 + y\left(\frac{t}{q(t)}\right)\right)^{k+1} = \frac{t}{q(t)}\cdot q(t)^{k+1}.$$

Therefore, $q(t) = 1 + tq(t)^k$. We set $tq(t^k) = b(t)$, then

$$b(t) = tq(t^k) = t\left(1 + t^k q(t^k)^k\right) = t(1 + b(t)^k).$$

This is in a form we apply Lagrange Inversion to get,

$$\begin{aligned}
[t^{n-r}]q(t)^n = [t^{kn-kr}]q(t^k)^n = [t^{kn-kr}]\left(\frac{b(t)}{t}\right)^n &= [t^{(k+1)n-kr}]b(t)^n \\
&= \frac{n}{(k+1)n-kr}[c^{kn-kr}](1+c^k)^{(k+1)n-kr} \\
&= \frac{n}{(k+1)n-kr}\binom{(k+1)n-kr}{n-r}.
\end{aligned}$$

So, the number of forests of k-noncrossing increasing trees with n vertices and r components is

$$\begin{aligned}
[x^n z^r]H(x,z) &= \frac{1}{n}\binom{n}{r-1}[t^{n-r}]q(t)^n = \frac{1}{n}\binom{n}{r-1}\frac{n}{(k+1)n-kr}\binom{(k+1)n-kr}{n-r} \\
&= \frac{1}{(k+1)n-kr}\binom{n}{r-1}\binom{(k+1)n-kr}{n-r}.
\end{aligned}$$

□

Setting $k = 2$ in (4.3), we obtain the formula for the number of forests of 2-noncrossing increasing trees with n vertices and r components such that the root of each tree is labelled by 2. This formula is given as

$$\frac{1}{3n-2r}\binom{n}{r-1}\binom{3n-2r}{n-r}.$$

Also, setting $k = 1$ in the same equation, we get the number of forests of noncrossing increasing trees on n vertices with r components as

$$\frac{1}{2n-r}\binom{n}{r-1}\binom{2n-r}{n-r} \tag{4.4}$$

and further on setting $r = 1$ in (4.4), we obtain the Catalan number which counts noncrossing increasing trees on n vertices.

5. Conclusion and Further work

In this paper, we have enumerated k-plane trees by root degree of a specified label. We also enumerated the trees by label of the first child of the root. Forests of k-plane trees are also enumerated in this paper. The work could be extended by enumerating forests of k-plane trees if degree sequences of all the vertices are taken into consideration. A study to delve into the number of k-plane trees based on the level of a vertex can also be done. It still remains an open problem to find an explicit formula for the number of k-plane trees with a given number of leaves.

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